

# Bose-Fermi Mixtures in One Dimension

Kunal K. Das\*

*Optical Sciences Center and Department of Physics, University of Arizona, Tucson, AZ 85721*

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We analyze the phase stability and the response of a mixture of bosons and spin-polarized fermions in one dimension (1D). Unlike in 3D, phase separation happens for low fermion densities. The dynamics of the mixture at low energy is independent of the spin-statistics of the components, and zero-sound-like modes exist that are essentially undamped.

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Binary mixtures of dilute quantum gases are a subject of steadily growing interest initiated by the realization of Bose Einstein condensation (BEC) of alkali atoms [1] and motivated by the quest for and subsequent experimental realization of degenerate Fermi gas [2]. Strong  $s$ -wave interactions which facilitate evaporative cooling of bosons are absent among spin polarized fermions due to the exclusion principle; so the method of choice for cooling fermions to degeneracy has been through the mediation of fermions in another spin state [2] or via a buffer gas of bosons [3, 4]. Degeneracy in dilute gases can be understood better than in their liquid helium counterparts due to the weaker interactions, and thus they offer prospects of detailed quantitative study of some of the most interesting phenomena in the physics of many body quantum systems such as the Bardeen-Cooper-Schrieffer (BCS) transition [5].

On another front a new generation of BEC experiments on surface micro-traps [6] and experiments on creating atomic waveguides [7] have generated interest in quantum gases in lower dimensions. Effective one and two dimensional BECs have been created, in which excitations in the confined directions are energetically not allowed [8]. Bose condensation on optical lattices [9] are being actively studied by several groups; the atoms at each lattice site can be in regimes of effective 1D.

It is therefore a natural step to bring these two exciting developments together and consider binary mixtures of quantum gases in effective 1D, with the possibility of forming one dimensional degenerate Fermi gases and fermionic waveguides. Fermions in one dimension have been the subject and the source of some seminal models in many-body quantum physics [10] mainly because they are theoretically more tractable than in 3D. Now there is actually the possibility of testing some of these models experimentally. Considerable recent theoretical work has already been done on three dimensional Bose-Fermi mixtures [11, 12, 13, 14, 15, 16, 17, 18, 19, 20] but very little has been said about one dimensional systems. The goal of this paper is to study theoretically some of the relevant properties of binary mixtures of bosons and spin polarized fermions in an effective one dimensional config-

uration. In particular we will consider their miscibility properties, phase stability and their excitations.

*Model:* We consider a longitudinally homogeneous one dimensional mixture of  $N_b$  hard-core bosons with mass  $m_b$  and  $N_f$  spin-polarized fermions with mass  $m_f$  at  $T = 0$  K. A natural choice of trap-geometry to consider such a mixture is in a toroidal trap [21, 22] with no external potential along the circumference (of length  $L$ ), but with tight cylindrically symmetric harmonic confinement of frequency  $\omega_0$  in the transverse direction. This geometry can equally well be interpreted as an infinitely long, straight waveguide with periodic boundary conditions. For the atoms to have effective 1D behavior at zero temperature the ground state energy of the transverse trapping potential has to be much higher than the ground state energy of the bosons and the fermions in 3D, i.e.  $\hbar\omega_0 \gg \mu_{b(3D)}$  and  $\hbar\omega_0 \gg \epsilon_{f(3D)}$  where  $\mu_{b(3D)} = 4\pi\hbar^2 a_b/m_b$  is the bosonic chemical potential and  $\epsilon_{f(3D)} = \hbar^2(6\pi^2 n_{f(3D)})^{2/3}/(2m_f)$  the Fermi energy of non-interacting fermions, both in 3D. A measure of the transverse spatial extent of the atoms in the ring is given by the single-particle ground state widths for the transverse trap,  $r_k = \sqrt{\hbar/m_k\omega_0}$  where  $k \rightarrow b, f$  and  $bf$  correspond to bosons, fermions and *twice* the reduced mass of a boson and a fermion  $m_{bf} = 2m_fm_b/(m_f + m_b)$ . A torus of high aspect ratio would have  $L \gg r_k$ .

Our treatment of the 1D Bose-Fermi mixture will rely on an effective Hamiltonian describing the longitudinal behavior of the gas in the toroidal trap

$$\hat{H} = \int dx \hat{\psi}_b^\dagger \left[ -\frac{\hbar^2}{2m_b} \partial_x^2 - \mu_b + \frac{g_b}{2} \hat{\psi}_b^\dagger \hat{\psi}_b \right] \hat{\psi}_b + \int dx \hat{\psi}_f^\dagger \left[ -\frac{\hbar^2}{2m_f} \partial_x^2 - \mu_f \right] \psi_f + g_{bf} \int dx \hat{\psi}_b^\dagger \hat{\psi}_b \psi_f^\dagger \psi_f \quad (1)$$

Here  $\hat{\psi}_b(x)$  and  $\hat{\psi}_f(x)$  are field operators for the longitudinal degree of freedom and  $x$  is the circumferential spatial coordinate. We have assumed factorization of the transverse degrees of freedom; such a factorization is justified in regimes of effective 1D [23] since the transverse spatial dependency is that of the single particle ground states  $\phi_{b0}(r)$  and  $\phi_{f0}(r)$  for the trapping potential regardless of the longitudinal behavior or statistics. Thus our effective 1D coupling strengths for the boson-boson

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\*Electronic address: kdas@optics.arizona.edu

and boson-fermion  $g_{bf}$  interactions are

$$g_b = \frac{4\pi\hbar^2 a_b}{m_b} \int 2\pi r dr |\phi_{b0}(r)|^4 = 2\hbar\omega_0 a_b$$

$$g_{bf} = \frac{4\pi\hbar^2 a_{bf}}{m_{bf}} \int 2\pi r dr |\phi_{b0}(r)|^2 |\phi_{f0}(r)|^2 = 2\hbar\omega_0 a_{bf} \quad (2)$$

with  $a_b$  and  $a_{bf}$  being the respective scattering lengths. The linear density operators are  $\hat{\rho}_b(x) = \hat{\psi}_b^\dagger(x)\hat{\psi}_b(x)$  and  $\hat{\rho}_f(x) = \hat{\psi}_f^\dagger(x)\hat{\psi}_f(x)$ , with spatially constant equilibrium expectations  $n_b = N_b/L$   $n_f = N_f/L$ ; the density fluctuation operators are therefore  $\delta\hat{\rho}_b(x) = \hat{\rho}_b(x) - n_b$  and  $\delta\hat{\rho}_f(x) = \hat{\rho}_f(x) - n_f$ .

*Phase stability in the static limit:* We first consider the mixture in static equilibrium in which case the expectations of the fluctuation operators are zero; the kinetic energy of the bosons vanishes while the kinetic energy for the fermions contributes the Fermi energy per particle  $\epsilon_f = g_f n_f^2/3$ , with  $g_f = \hbar^2 \pi^2 / (2m_f)$  and the Fermi wave-vector  $k_f = \pi n_f$ . In this static case the *total* number of particles is fixed, so we take the ground state expectation of the *canonical* Hamiltonian for the system and thus obtain a simple expression for the total energy of a *uniform* mixture of bosons and fermions at equilibrium

$$E_u = L \left[ \frac{g_b}{2} n_b^2 + \frac{g_f}{3} n_f^3 + g_{bf} n_b n_f \right]. \quad (3)$$

The first derivative of this with respect to the densities yield the Thomas-Fermi equations

$$\mu_b = g_b n_b + g_{bf} n_f \quad \mu_f = g_{bf} n_b + g_f n_f^2, \quad (4)$$

for the chemical potentials, and the derivative with respect to the linear-volume ( $L$ ) gives the pressure

$$p = -\frac{\partial E}{\partial L} = \frac{g_b}{2} n_b^2 + \frac{2g_f}{3} n_f^3 + g_{bf} n_b n_f. \quad (5)$$

The second derivative condition for a stable minimum with respect to small changes in the densities puts a *lower limit* on the fermion density

$$n_f \geq \frac{g_{bf}^2}{2g_f g_b} = \frac{2}{\pi^2} \frac{a_{bf}^2}{a_b r_f^2}. \quad (6)$$

This constraint is the opposite of that in 3D where the stability condition puts an *upper limit* on the fermion density. The reason for the difference is that the power law of the density dependency of the Fermi energy changes with dimensionality. The energy contribution from the Fermi pressure grows faster as a function of linear density in 1D than it does with increase in bulk density in 3D, however the boson-fermion interaction energy behaves similarly in 3D and in 1D with respect to bulk density and linear density respectively; thus at higher fermion densities, the total energy in 1D is more likely to be lowered if the fermions are spread out over a larger volume mixed in with the bosons.

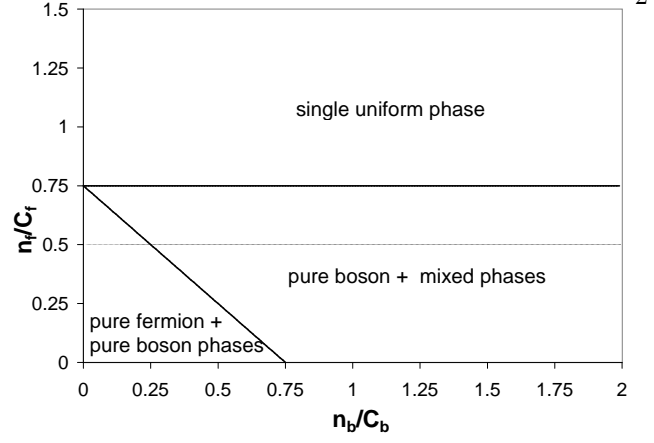


FIG. 1: Phase diagram for a mixture of bosons and fermions in one dimension. The thin line corresponds to the linear stability condition in Eq. (6).

The stability criterion in (6) applies for small fluctuations, we now analyze the general phase stability for a Bose-Fermi mixture in 1D; such an analysis was done for mixtures in 3D by Viverit *et al.* [13]. A binary mixture can have at most two distinct phases that we label  $i = 1, 2$ . The linear volume occupied by each phase is  $L_i$ , the number of bosons (fermions) therein  $N_{b(f),i}$  and the corresponding densities  $n_{b(f),i} = N_{b(f),i}/L_i$ . The volume fractions of the phases are  $\ell = L_1/L$  and  $1 - \ell = L_2/L$  and ratio of the densities in the two phases are labelled  $\eta_{b(f)} = n_{b(f),1}/n_{b(f),2}$ . The total energy for the phase-separated mixture is

$$E_s = \sum_{i=1}^2 L_i E_i = \sum_{i=1}^2 L_i \left[ \frac{g_b}{2} n_{b,i}^2 + \frac{g_f}{3} n_{f,i}^3 + g_{bf} n_{b,i} n_{f,i} \right] \quad (7)$$

Equilibrium between the phases require

$$(a) \quad p_1 = p_2; \quad (b) \quad \mu_{b(f),1} = \mu_{b(f),2} \\ (c) \quad \mu_{b(f),i} > \mu_{b(f),j} \text{ if } n_{b(f),i} = 0, \quad (8)$$

where the pressure and chemical potentials in each phase are given by Eqs. (4) and (5) with the total densities *replaced* by partial densities. We will use the identity  $\ell n_{b(f),1} + (1 - \ell) n_{b(f),2} = n_{b(f)}$  in Eq. (3) to evaluate the energy  $E_u$  of the uniform phase to compare with the energy  $E_s$  of the phase-separated mixture in Eq. (7). It is convenient to introduce density measures in terms of the interaction strengths:  $C_f = g_{bf}^2 / (g_f g_b)$  for fermions and  $C_b = g_{bf}^3 / (g_f g_b^2)$  for bosons. There four possible ways of phase separation, the feasibility of each is determined by the specific nature of the conditions (8) and the principle of minimum energy. We now discuss each case, leaving out the somewhat tedious algebra for brevity.

(i) Two pure phases : The fermions are all in one phase and the bosons in the other, so we set  $n_{f,1} = n_{b,2} = 0$ . The equilibrium conditions (8a) and (8c) constrain the partial densities:  $n_{f,2} \leq 3C_f/4$  and  $n_{b,1} \leq 3C_b/4$ . When those conditions are used in Eqs. (3) and (7) they give

$$\frac{1}{L} [E_u - E_s] \geq (1 - \ell) \ell^2 \frac{g_f}{3} n_{f,2}^3 \geq 0, \quad (9)$$

which means that the separated phase has lower energy for all values of the volume fraction  $\ell \in [0, 1]$  and hence is energetically preferred in the density regimes where phase equilibrium is possible:

$$n_f \leq (1 - \ell) \frac{3}{4} C_f \quad n_b \leq \ell \frac{3}{4} C_b. \quad (10)$$

(ii) A mixed phase and purely bosonic phase : The fermions are all in one phase,  $n_{f,1} = 0$  but there are bosons in both phases. The two equations arising from pressure equality (8a) and the equality of boson chemical potentials (8b) fix the fermion partial density  $n_{f,2} = 3C_f/4$  which obeys the condition  $n_{f,2} \leq C_f$  due to the inequality of the fermion chemical potentials  $\mu_{f,1} \geq \mu_{f,2}$ . Then it follows from  $\mu_{b,1} = \mu_{b,2}$  that  $n_{b,1} - n_{b,2} = 3C_b/4$ .

On applying the equations for the partial pressures and the boson chemical potentials to Eqs. (3) and (7) we find

$$\frac{1}{L} [E_u - E_s] = (1 - \ell) \ell^2 \frac{g_F}{3} n_{F,2}^3 \geq 0. \quad (11)$$

so that the separated phase is energetically preferred in this case as well, in the density regimes given by

$$n_f = (1 - \ell) \frac{3C_f}{4} \quad n_b = n_{b,2} + \ell \frac{3C_b}{4} \quad (12)$$

that do not overlap with those of the previous case.

(iii) A mixed phase and purely fermionic one : All the bosons are in one phase,  $n_{b,2} = 0$  while the fermions can be in both phases. The phase equilibrium conditions require that the fermion density ratio  $\eta_f \in (0, 1)$  and that it satisfies the equation

$$\frac{3n_{f,2}}{2C_f} (1 + \eta_f)^2 = (2 + \eta_f). \quad (13)$$

This limits the fermion density in the second phase to be  $C_f/2 \leq n_{f,2} \leq 4C_f/3$ . However on using the equality of partial pressures (8a) and that of the fermion chemical potentials in the two phases,  $\mu_{f,1} = \mu_{f,2}$  we find that

$$\frac{1}{L} [E_u - E_s] = \ell(1 - \ell)^2 \frac{g_F}{3} [n_{F,1} - n_{F,2}]^3 \leq 0, \quad (14)$$

where the inequality holds because  $\eta_f \leq 1$ . The uniform mixture will thus be energetically preferred for all values of  $\ell$ , hence this phase separation will not occur.

(iv) Two mixed phases : Finally we consider the case where both phases have fermions as well as bosons. There are three equations arising from the equilibrium conditions (8a) and (8b) which lead to the following equation for the fermion density ratio

$$(1 - \eta_f)^3 = 0 \quad (15)$$

with solution  $\eta_f = 1$  which, due to  $\mu_{b,1} = \mu_{b,2}$ , also implies that the boson density ratio  $\eta_b = 1$  which means that the only allowed solution is when the entire system is uniform, and there is no phase separation of this type.

The allowed phases for different linear densities are plotted in Fig. (1); phase separation occurs for low fermion densities, in qualitative agreement with the linear stability condition. Taking the bulk density to be  $n_{f(3D)} \simeq n_f/\pi r_f^2$ , the criteria for single phase  $n_f > \frac{3}{4}C_f$  and effective one-dimensionality  $\epsilon_{f(3D)} \ll \hbar\omega_0$  give the limits of fermion density for which bosons and fermions in 1D can coexist in a single phase:  $a_{bf}^2/(a_b r_f^2) < n_f < 1/r_f$ . Transverse trap widths  $r_f \sim 1 \mu\text{m}$  achievable currently would allow single phase mixtures for fermion bulk densities up to  $n_{f(3D)} \sim 10^{18} \text{ m}^{-3}$ , which corresponds to the density of the coldest  $^6\text{Li}$  samples in a recent experiment [24] that created a degenerate system of bosons ( $^{23}\text{Na}$ ) and fermions ( $^6\text{Li}$ ) in 3D. Scattering lengths for alkali atoms are of the order of  $\sim 1 - 10 \text{ nm}$ , that would allow a range of few orders of magnitude of fermion density where a stable uniform 1D mixture of bosons and fermions would form; this range can be widened by increasing the transverse trap strength or reducing the boson-fermion scattering length. Unlike in 3D, phase separation effects in 1D can be observed by reducing the density which is usually easier to do than increasing it.

*Dynamic response:* We now consider the dynamical properties of the mixture. For weak interaction strengths and low energy modes we can use linear response theory which is a convenient formulation of first order time-dependent perturbation theory in the interaction picture. We consider small density fluctuations of the bosons and the fermions about equilibrium. The boson fluctuation can be considered a density dependent perturbation for the fermions and vice versa, so that we have two coupled linear equations for the expectation of the density fluctuations  $\delta\rho_b(x)$  and  $\delta\rho_f(x)$

$$\begin{aligned} \delta\rho_b(q, \omega) &= \chi_b \cdot g_{bf} \delta\rho_f(q, \omega) \\ \delta\rho_f(q, \omega) &= \chi_f \cdot g_{fb} \delta\rho_b(q, \omega) \end{aligned} \quad (16)$$

with retarded density-density response functions

$$\chi = \frac{1}{\hbar} \sum_{n \neq 0} |\langle n | \delta\rho^\dagger(\mathbf{k}) | 0 \rangle|^2 \left[ \frac{2\omega_{n0}}{(\omega + i\eta)^2 - \omega_{n0}^2} \right]. \quad (17)$$

The small imaginary shift  $i\eta$  preserves causality and the ground state  $|0\rangle$  represents the Fermi sea for the fermions and the condensate for bosons. In the Bogoliubov approximation for the bosons the response function as well as the quasi-particle spectrum in 1D have algebraic forms identical to those in 3D and are given by

$$\begin{aligned} \chi_b(q, \omega) &= \frac{n_b q^2}{m_b [\omega^2 - \omega_b^2(q)]} \\ \omega_b(q)^2 &= (\epsilon_q/\hbar)^2 + (v_b q)^2 \end{aligned} \quad (18)$$

with free quasiparticle energy  $\epsilon_q = \hbar^2 q^2/(2m_b)$  and sound velocity  $v_b = \sqrt{g_b n_b/m_b}$ . The poles correspond to the energies of the collective modes which are undamped in the Bogoliubov approximation.

The fermions being spin-polarized do not have s-wave interaction so that the response function for the fermions is taken to be that for free fermions; unlike the bosonic response function this has a form in 1D quite distinct from that in 3D:

$$\chi_f(q, \omega) = \frac{m_f}{2\pi\hbar^2 q} \ln \left[ \frac{(\omega + i\eta)^2 - \omega_-^2}{(\omega + i\eta)^2 - \omega_+^2} \right]$$

$$\hbar^2 \omega_\pm^2 = \frac{\hbar^4}{4m_f^2} ((k_f \pm q)^2 - k_f^2)^2 \quad (19)$$

The calculation leading from the general expression Eq. (17) to Eq. (19) is analogous to that in 3D [25] with the Fermi-sphere replaced by a “Fermi-interval”  $[-k_F, k_F]$ . It is apparent that  $\text{Im } \chi_f(q, \omega) \neq 0$  only if  $|\omega_-| \leq |\omega| \leq |\omega_+|$ .

The low energy, long wavelength collective modes are of particular experimental interest, so we do a Taylor expansion of the expression (19) for the fermionic response function  $\chi_f$  for small values of  $q$ , but keeping the ratio of the energy transfer to momentum transfer  $\omega/q$  constant. The result is quite interesting

$$\chi_f(q, \omega) \simeq \frac{n_f q^2}{m_f [(\omega + i\eta)^2 - \omega_f(q)^2]}$$

$$\omega_f(q)^2 = (\epsilon_q/\hbar)^2 + (v_f q)^2. \quad (20)$$

It is apparent that if we replace the Bogoliubov sound velocity with the Fermi velocity  $v_b \rightarrow v_f = \hbar k_f/m$ , the real part of this limiting form is *identical* to the Bogoliubov density-density response function; and the spectrum corresponding to its poles are identical in form with that of the Bogoliubov poles.

This equivalence of the bosonic and fermionic density fluctuations is distinctly a property of one dimension with no analog in higher dimensions. Such an equivalence is not surprising when one recalls the Luttinger liquid model of Haldane [26] where the low energy behavior of quantum fluids in one dimension were shown to be independent of spin-statistics. However it is important to note that the fermionic response function that we consider above is that for free fermions while the Luttinger-Tomonaga [10] model assumes long range interactions among the fermions.

The absence of interaction among fermions distinguishes the qualitative nature of the fermionic excitations from the Bogoliubov modes, despite the similarity in the low energy structure of the response functions. Strictly speaking the fermionic excitations are elementary excitations, whereas the Bogoliubov modes are collective modes of the bosons for which the interactions play a central role. One could of course interpret the fermionic excitations as zero sound modes in the limit of vanishing interaction; in that limit we recall that the zero sound velocity coincides with the Fermi velocity even in 3D.

The similarity of the response functions at low energies combined with the fact that homogeneous Bogoliubov modes have the same form in 1D and 3D allows us

to directly apply the results obtained for spatially uniform binary mixtures of bosons in three dimensions to the Bose-Fermi mixtures in 1D. As an example, we consider the normal modes of the mixture determined by the vanishing of the coefficient determinant of the response equations (16),  $1 - g_{bf}^2 \chi_b \chi_f = 0$ , which leads to an expression for the normal mode velocities similar to that for binary mixtures of bosons in 3D

$$v_\pm^2 = \frac{1}{2} \left[ (v_b^2 + v_f^2) \pm \sqrt{(v_b - v_f)^2 + 4g_{bf}^2 \frac{n_f n_b}{m_f m_b}} \right]. \quad (21)$$

Here we have used the linear dispersions  $\omega_b(q) \simeq v_b q$  and  $\omega_f(q) \simeq v_f q$  for long wavelength modes. Hydrodynamic equations for boson-fermion density fluctuations in the collisional regime also give a similar expression for the sound velocities [14], but with the crucial difference that the fermion sound velocity in that case is that of first sound. In the static limit  $q \rightarrow 0$  we find that the condition  $v_\pm^2 \geq 0$ , necessary for positive compressibility, leads to the same condition obtained earlier in Eq. (6) from energy considerations.

The frequencies corresponding to the original low energy Bogoliubov phonons shift due to the interaction with the fermions by about  $\delta\omega = \omega_q - v_b q \simeq n_f n_b g_{bf}^2 q^2 / m_b m_f (v_b^2 - v_f^2)$ ; the shift is positive or negative depending on whether  $v_b > v_f$  or  $v_b < v_f$ , similar to the behavior in 3D [15]. But these modes in the mixed system differ from their analog in 3D mixtures in that they are not damped unless the velocity matches the Fermi velocity as seen from Eq. (20). In 3D mixtures, such modes are damped if  $v_b < v_f$ .

For modes with higher momentum where the exact fermion response function (19) has to be used, there is damping for mode frequencies in the range  $|\omega_-| \leq |\omega| \leq |\omega_+|$  with the damping rate given to lowest order by  $\gamma \sim m c_b g_{bf}^2 / (4\hbar^2 g_b)$ . The key difference from 3D is that this rate is *independent* of the mode.

The exact fermionic response function (19) in one dimension has several interesting features distinct from 3D. For zero energy transfer it is seen that  $\chi_f(q, 0)$  has a logarithmic divergence at  $q = 2k_f$  due to perfect nesting, whereas in 3D the derivative of the response function is divergent, which leads to Friedel oscillations. In 1D Bose-Fermi mixtures, the logarithmic divergence of response function leads to periodic density variations in the fermions of period  $2k_f$  associated with the formation of coherent superposition of particle-hole pair states called the Peierls channel [27]. Due to the boson-fermion density coupling, the bosons acquire a similar periodicity but out of phase with the fermion density modulation as demonstrated variationally by Miyakawa *et al.* [28].

In conclusion, we have studied the phase stability of a boson-fermion mixture in one dimension and demonstrated that phase separation would occur at low fermion densities a behavior opposite to that in 3D. This means that phase separation effects may be studied at densities easier to achieve than in 3D. The regimes of co-existence of bosons and fermions in the same space are

within the reach of experimental capabilities, and there is the exciting prospect of creating degenerate fermions in one dimension. Also we have shown that the low energy density-density response of free fermions is identical in form to that of weakly interacting bosons; this means that binary mixtures will have similar normal modes regardless of whether the components are bosons or fermions. These modes are analogous to zero sound modes and are essentially undamped at low momenta. Away from the low energy regime, the similarity of re-

sponse functions does not hold, and the fermionic response in 1D acquires interesting features, in particular a logarithmic divergence at twice the Fermi momentum which leads to periodic density modulation of the system analogous to Friedel oscillations.

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